

REPRESENTATION THEORY OF THE VERTEX ALGEBRA $W_{1+\infty}$

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ABSTRACT

In our paper [KR] we began a systematic study of representations of the universal central extension $\hat{\mathcal{D}}$ of the Lie algebra of differential operators on the circle. This study was continued in the paper [FKRW] in the framework of vertex algebra theory. It was shown that the associated to $\hat{\mathcal{D}}$ simple vertex algebra $W_{1+\infty,N}$ with positive integral central charge N is isomorphic to the classical vertex algebra $W(gl_N)$, which led to a classification of modules over $W_{1+\infty,N}$. In the present paper we study the remaining non-trivial case, that of a negative central charge $-N$. The basic tool is the decomposition of N pairs of free charged bosons with respect to gl_N and the commuting with gl_N Lie algebra of infinite matrices \hat{gl} .

INTRODUCTION

In this paper we study representations of a central extension $\hat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}C$ of the Lie algebra \mathcal{D} of differential operators on the circle. This central extension appeared first in [KP] and its uniqueness was subsequently established in [F] and in [Li].

In our paper [KR] we began a systematic study of representations of the Lie algebra $\hat{\mathcal{D}}$. In particular, we classified the irreducible “quasi-finite” highest weight representations and constructed them in terms of irreducible highest weight representations of the central extension \hat{gl} of the Lie algebra of infinite matrices.

This study was continued in [FKRW] in the framework of vertex algebra theory. The advantage of such an approach is twofold. From the mathematical point of view it picks out the most interesting representations and equips them with a rich additional structure. From the physics point of view it provides building blocks of two-dimensional conformal field theories. We just mention here applications to integrable systems [ASM], to W -gravity [BMP], and to the quantum Hall effect [CTZ1,CTZ2].

In more detail, let us consider the subalgebra \mathcal{P} of \mathcal{D} consisting of the differential operators that can be extended in the interior of the circle. The cocycle of our central extension (see (6.2)) restricts to a zero cocycle on \mathcal{P} , hence \mathcal{P} is a subalgebra of $\hat{\mathcal{D}}$. Given $c \in \mathbb{C}$, we denote by M_c the representation of $\hat{\mathcal{D}}$ induced from the 1-dimensional representation of the subalgebra $\mathcal{P} \oplus \mathbb{C}C$ defined by $\mathcal{P} \mapsto 0$, $C \mapsto c$. It was shown in [FKRW] that M_c carries a canonical vertex algebra structure. The

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$\hat{\mathcal{D}}$ -module M_c has a unique irreducible quotient which carries the induced simple vertex algebra structure. Following physicists, we denote this simple vertex algebra by $W_{1+\infty,c}$.

The highest weight representations of the vertex algebra M_c are in a canonical 1–1 correspondence with that of the Lie algebra $\hat{\mathcal{D}}$. As usual, the situation is much more interesting when we pass to the simple vertex algebra $W_{1+\infty,c}$. The cocycle (6.2) is normalized in such a way that M_c is an irreducible $\hat{\mathcal{D}}$ -module (*i.e.*, $M_c = W_{1+\infty,c}$) iff $c \notin \mathbb{Z}$.

Thus, we are led to the problem of classification of irreducible highest weight representations of the vertex algebra $W_{1+\infty,c}$ with $c \in \mathbb{Z}$. This is a highly non-trivial problem since the field corresponding to every singular vector of M_c must vanish in a representation of $W_{1+\infty,c}$ which gives rise to an infinite set of equations.

The problem has been solved in [FKRW] in the case of a positive integral $c = N$ by the use of an explicit isomorphism of $W_{1+\infty,N}$ and the classical W -algebra $W(gl_N)$. This approach cannot be used for c negative since probably $W_{1+\infty,-N}$ is a new structure, which is not isomorphic to any classical vertex algebra.

Another important result of [FKRW] is an explicit construction of a “model” of “integral” representations of $W_{1+\infty,N}$ by decomposing the vertex algebra of N charged free fermions with respect to the commuting pair of Lie algebras gl_N and \widehat{gl} .

One of the main results of this paper is an analogous decomposition of the vertex algebra of N charged free bosons with respect to a commuting pair gl_N and \widehat{gl} , which produces a large class of irreducible modules of the vertex algebra $W_{1+\infty,-N}$ (Theorems 3.1 and 6.1). We conjecture that all irreducible modules of $W_{1+\infty,-N}$ can be obtained from these by restricting to $\hat{\mathcal{D}}$ and taking their certain tensor products (Conjecture 6.1). This explicit construction allows one to also derive explicit character formulas (formula (3.7)). Partial results in this direction were previously obtained in [Mat] and [AFMO].

Our basic tool is the suitably modified theory of dual pairs of Howe [H1], [H2]. Namely one has the following general irreducibility theorem (Theorem 1.1): if a Lie algebra \mathfrak{g} acts completely reducibly on an associative algebra A and if V is an irreducible A -module with an equivariant action of \mathfrak{g} such that V is a direct sum of at most countable number of irreducible finite-dimensional modules, then the pair \mathfrak{g} and $A^{\mathfrak{g}} = \{a \in A \mid \mathfrak{g}a = 0\}$ acts irreducibly on each isotypic component of \mathfrak{g} in V .

This result allows us not only to decompose both free charged fermions and bosons with respect to the pair gl_N and \widehat{gl} , but also to interpret the vertex algebra $W_{1+\infty,-N}$ as a subalgebra of the vertex algebra of N free charged bosons killed by gl_N (formula (4.4)). In the same way, one identifies $W_{1+\infty,N}$ with a subalgebra of the vertex algebra of N free charged fermions killed by gl_N , a result previously obtained in [FKRW]. This result is interpreted in that paper as an isomorphism $W_{1+\infty,N} \simeq W(gl_N)$ due to the connection to affine Kac-Moody algebras. It seems, however, that there is no such interpretation in the case of negative central charge.

It is interesting to note that while the decomposition of N free charged fermions produces all unitary (with respect to the compact involution) irreducible highest weight representations of \widehat{gl} with central charge N , the decomposition of N free charged bosons produces a very interesting class of irreducible highest weight representations of \widehat{gl} with central charge $-N$. This allows us to compute their characters (formula (3.7)). We also show that the subcategory of the category \mathcal{O} of representations all of whose irreducible subquotients are members of this class is a semisimple

category (Theorem 4.1). (Of course, in the first case a similar result goes back to H. Weyl.) Provided that Conjecture 6.1 is valid, this implies semisimplicity of the category of positive energy $W_{1+\infty, -N}$ -modules.

The paper is organized as follows. In Section 1 we give a proof of the general irreducibility theorem (Theorem 1.1). In Section 2 we use Theorem 1.1 and classical invariant theory (as in [H2]) to prove irreducibility of each isotypic component of gl_N in the metaplectic representation of the infinite Weyl algebra W_N (N free charged bosons in physics terminology) with respect to the commuting pair gl_N and \widehat{gl} (Theorem 2.1). By a somewhat lengthy combinatorial argument, we derive an explicit highest weight correspondence in Section 3 (Theorem 3.1) and a character formula for the above-mentioned highest weight representations of \widehat{gl} (formula (3.7)). In Section 4 we prove the complete reducibility Theorem 4.1. Section 5 is a brief digression on vertex algebras and their twisted modules which we conclude by a construction of twisted modules over N free charged bosons which become untwisted modules with respect to $W_{1+\infty, -N}$. In Section 6 we construct a large family of representations of the vertex algebra $W_{1+\infty, -N}$ using the above mentioned modules (Theorem 6.2) and conjecture that these are all its irreducible representations (Conjecture 6.1). In Section 7 we apply similar methods to N free charged fermions to recover most of the results of [FKRW].

1. REPRESENTATIONS OF ASSOCIATIVE \mathfrak{g} -ALGEBRAS

Let A be an associative algebra over \mathbb{C} and let $\text{Der } A$ denote the Lie algebra of derivations of A . Let \mathfrak{g} be a Lie algebra over \mathbb{C} and let $\varphi : \mathfrak{g} \rightarrow \text{Der } A$ be a Lie algebra homomorphism. The triple $(A, \mathfrak{g}, \varphi)$ is called an *associative \mathfrak{g} -algebra*.

An A -module V is called a (\mathfrak{g}, A) -module if V is given a structure of a \mathfrak{g} -module such that the A -module structure is equivariant, i.e.

$$g(av) = (\varphi(g)a)v + a(gv), \quad g \in \mathfrak{g}, a \in A, v \in V.$$

Let $A^{\mathfrak{g}} = \{a \in A \mid \varphi(\mathfrak{g})a = 0 \text{ for all } g \in \mathfrak{g}\}$ be the centralizer of the action of \mathfrak{g} on A . Given a \mathfrak{g} -submodule U of V and $a \in A^{\mathfrak{g}}$, the map $U \rightarrow aU$, given by $u \mapsto au$, is clearly a \mathfrak{g} -module homomorphism.

Given an irreducible \mathfrak{g} -module E , denote by V_E the sum of all \mathfrak{g} -submodules of V isomorphic to E . This is called the *E -isotypic component* of the \mathfrak{g} -module V . By the above remark, V_E is a $A^{\mathfrak{g}}$ -submodule of V .

Choose a 1-dimensional subspace $f \subset E$. Then, due to Schur's lemma, this gives us a choice of a 1-dimensional subspace in each of the irreducible \mathfrak{g} -submodules of V_E . We denote the sum of all of these 1-dimensional subspaces by V^E (it depends on the choice of f). Clearly, V^E is a $A^{\mathfrak{g}}$ -submodule of V_E , and we have a (non-canonical) $(\mathfrak{g}, A^{\mathfrak{g}})$ -module isomorphism:

$$V_E \simeq E \otimes V^E.$$

The Lie algebra \mathfrak{g} (resp. associative algebra $A^{\mathfrak{g}}$) acts on $E \otimes V^E$ by $g(e \otimes v) = ge \otimes v$ (resp. $a(e \otimes v) = e \otimes av$).

Thus, if V is a (\mathfrak{g}, A) -module, which is a semisimple \mathfrak{g} -module, we have the following isomorphism of $(\mathfrak{g}, A^{\mathfrak{g}})$ -modules:

$$(1.1) \quad V \simeq \bigoplus_E (E \otimes V^E)$$

where summation is taken over all equivalence classes of irreducible \mathfrak{g} -modules.

We wish to study the situations when each isotypic component V_E is irreducible as a $(\mathfrak{g}, A^\mathfrak{g})$ -module, or equivalently, when each $A^\mathfrak{g}$ -module V^E is irreducible.

Theorem 1.1. *Let A be a semisimple \mathfrak{g} -module. Let V be a (\mathfrak{g}, A) -module such that*

- (i) *V is an irreducible A -module*
- (ii) *V is a direct sum of at most countable number of finite-dimensional irreducible \mathfrak{g} -modules.*

Then each isotypic component V_E (where E is a finite-dimensional irreducible \mathfrak{g} -module) is an irreducible $(\mathfrak{g}, A^\mathfrak{g})$ -module; equivalently, each V^E is an irreducible $A^\mathfrak{g}$ -module.

The proof of the theorem is based on some simple lemmas.

Lemma 1.1 (cf. [H2]). *Each V^E is an irreducible $(\text{End } V)^\mathfrak{g}$ -module. (Recall that \mathfrak{g} acts on $\text{End } V$ by $(ga)v = g(av) - a(gv)$, $g \in \mathfrak{g}$, $a \in \text{End } V$, $v \in V$.)*

Proof. Let $x, y \in V^E$. We have the following decomposition in a direct sum of \mathfrak{g} -modules:

$$V = (E \otimes x) \oplus (E \otimes y) \oplus \mathcal{U}.$$

Then $a \in \text{End } V$ defined by $(f_i \in E, u \in \mathcal{U})$:

$$a(f_1 \otimes x + f_2 \otimes y + u) = f_1 \otimes y + f_2 \otimes x + u$$

lies in $(\text{End } V)^\mathfrak{g}$. This proves the lemma. \square

Lemma 1.2 (cf. [H2]). *Let X be a finite-dimensional \mathfrak{g} -invariant subspace of V . Then the \mathfrak{g} -module $\text{Hom}_\mathbb{C}(X, V)$ is semisimple and the \mathfrak{g} -module map $A \rightarrow \text{Hom}_\mathbb{C}(X, V)$ (defined by $a \mapsto \varphi_a$ where $\varphi_a(x) = ax$, $x \in X$) commutes with projections on isotypic components.*

Proof. Let $U = \text{Hom}_\mathbb{C}(X, V)$. Since $\dim X < \infty$, we have the \mathfrak{g} -module isomorphism $U \simeq X^* \otimes V$, hence U is a semisimple \mathfrak{g} -module:

$$U = \bigoplus_E (E \otimes U^E).$$

Since A is a semisimple \mathfrak{g} -module, we have:

$$A = \bigoplus_E (E \otimes A^E).$$

Let $A_X = \{a \in A \mid aX = 0\}$. This is a \mathfrak{g} -submodule of A , so that

$$A_X = \bigoplus_E (E \otimes A_X^E), \quad A_X^E \subset A^E,$$

and, clearly the map $A \rightarrow U$ coincides with the canonical map

$$\bigoplus (E \otimes A^E) \rightarrow \bigoplus (E \otimes (A^E / A_X^E)).$$

This proves the lemma. \square

Lemma 1.3. (*Jacobson's density theorem*) *Let V be at most countable-dimensional irreducible A -module. Then for any finite-dimensional subspace X of V and any $f \in \text{End}_{\mathbb{C}} V$ there exists $a \in A$ such that*

$$f(x) = a(x) \text{ for all } x \in X.$$

Proof. $\text{End}_A V = \mathbb{C}$ since V is irreducible and at most countable-dimensional. Now lemma follows from Jacobson's density theorem as stated in [L]. \square

Proof of Theorem 1.1. Let $x, y \in V^E$. By Lemma 1.1 there exists $f \in (\text{End } V)^{\mathfrak{g}}$ such that $f(x) = y$. By Lemma 1.3 there exists $a \in A$ such that a coincides with f on the subspace $U = E \otimes x + E \otimes y$. Let a_0 be the projection of a on $A^{\mathfrak{g}}$. By Lemma 1.2, a_0 still coincides with f on U . Indeed, we have for $u \in U$, the subscript E denoting the projection on V_E :

$$\alpha_0(u) = a(u)_E = f(u)_E = f(u). \quad \square$$

We do not know whether one can remove the finite-dimensionality assumption in (ii). The following proposition shows that one can for the isotypic component of the trivial \mathfrak{g} -module.

Proposition 1.1. *Let A be an associative \mathfrak{g} -algebra, completely reducible with respect to \mathfrak{g} . Let V be an (A, \mathfrak{g}) -module which is irreducible with respect to A and completely reducible with respect to \mathfrak{g} . Then $V^{\mathfrak{g}} := \{v \in V \mid \mathfrak{g}v = 0\}$ is an irreducible $A^{\mathfrak{g}}$ -module.*

Proof. We have the decompositions into isotypic components with respect to \mathfrak{g} :

$$A = A^{\mathfrak{g}} \oplus \sum_{E \neq 1} A_E, \quad V = V^{\mathfrak{g}} \oplus \sum_{E \neq 1} V_E,$$

where 1 stands for the 1-dimensional trivial representation of \mathfrak{g} . Note that

$$\left(\sum_{E \neq 1} A_E \right) V^{\mathfrak{g}} \subset \sum_{E \neq 1} V_E, \quad A^{\mathfrak{g}} V^{\mathfrak{g}} \subset V^{\mathfrak{g}}.$$

Let $v \in V^{\mathfrak{g}}$ be a non-zero vector. Since V is an irreducible A -module, we have: $Av = V$. Hence:

$$V = Av = A^{\mathfrak{g}}v \oplus \left(\sum_{E \neq 1} A_E \right) v \subset A^{\mathfrak{g}}v \oplus \sum_{E \neq 1} V_E.$$

It follows that $A^{\mathfrak{g}}v = V^{\mathfrak{g}}$. \square

Remark 1.1. Theorem 1.1 and Proposition 1.1 have obvious group analogues. Let G be a group and let $\varphi : G \rightarrow \text{Aut } A$ be a group homomorphism. This is called an associative G -algebra. An A -module V is called a (G, A) -module if V is given a structure of a G -module such that the A -module structure is G -equivariant, i.e.

$$g(av) = (\varphi(g)a)(gv), \quad g \in G, a \in A, v \in V.$$

Then Theorem 1.1 and Proposition 1.1 hold with \mathfrak{g} replaced by G .

2. CHARGED FREE BOSONS

Consider N pairs of free charged bosonic fields ($i = 1, \dots, N$):

$$\gamma^i(z) = \sum_{m \in \mathbb{Z}} \gamma_m^i z^{-m-1}, \quad \gamma^{*i}(z) = \sum_{m \in \mathbb{Z}} \gamma_m^{*i} z^{-m}.$$

Recall that this is a collection of local even fields with the OPE ($i, j = 1, \dots, N$):

$$(2.1) \quad \gamma^{*i}(z) \gamma^j(w) \sim \frac{\delta_{ij}}{z-w}, \quad \text{all other OPE} \sim 0,$$

with the vacuum vector $|0\rangle$ subject to conditions

$$(2.2) \quad \gamma_m^i |0\rangle = 0 \text{ for } m \geq 0, \quad \gamma_m^{*i} |0\rangle = 0 \text{ for } m > 0.$$

In other words, we have a unital associative algebra, usually called the *Weyl algebra* and denoted by W_N , on generators $\gamma_m^i, \gamma_m^{*i}$ ($i = 1, \dots, N; m \in \mathbb{Z}$) with the following defining relations:

$$(2.3) \quad [\gamma_m^{*i}, \gamma_n^j] = \delta_{ij} \delta_{m,-n} \quad [\gamma_m^i, \gamma_n^j] = 0 = [\gamma_m^{*i}, \gamma_n^{*j}].$$

This algebra has a unique irreducible representation in a vector space M , called the *metaplectic representation*, such that there exists a non-zero vector $|0\rangle$ satisfying conditions (2.2).

We let

$$e^{ij}(z) := \gamma^i(z) \gamma^{*j}(z) \equiv \sum_{m \in \mathbb{Z}} e_m^{ij} z^{-m-1} \quad (i, j = 1, \dots, N),$$

$$E(z, w) = \sum_{p=1}^N : \gamma^p(z) \gamma^{*p}(w) : \equiv \sum_{i,j \in \mathbb{Z}} E_{ij} z^{i-1} w^{-j},$$

where the normal ordering $::$ as usual means that the operators annihilating $|0\rangle$ are moved to the right.

Using Wick's formula, it is immediate to see that the operators e_m^{ij} form a representation in M of the affine Kac-Moody algebra \widehat{gl}_N of level -1 [K1]:

$$(2.4) \quad [e_m^{ij}, e_n^{st}] = \delta_{js} e_{m+n}^{it} - \delta_{it} e_{m+n}^{sj} - m \delta_{m,-n} \delta_{js} \delta_{it}.$$

In particular, the operators $e_{ij} := e_0^{ij}$ form a representation of the general linear Lie algebra gl_N . We have also the following commutation relations:

$$(2.5) \quad [e_m^{ij}, \gamma_n^k] = \delta_{jk} \gamma_{m+n}^i, \quad [e_m^{ij}, \gamma_n^{*k}] = -\delta_{ik} \gamma_{m+n}^j.$$

Hence the Lie algebra \widehat{gl}_N acts on the Weyl algebra W_N by derivations via the adjoint representation $g(a) = [g, a]$.

The following important relation is straightforward:

$$(2.6) \quad [e_{ij}, E_{mn}] = 0 \quad \text{for all } i, j = 1, \dots, N; m, n \in \mathbb{Z}.$$

Furthermore, the operators E_{ij} ($i, j \in \mathbb{Z}$) form a representation in M of the Lie algebra \widehat{gl} with central charge $-N$:

$$(2.7) \quad [E_{ij}, E_{st}] = \delta_{js} E_{it} - \delta_{it} E_{sj} - N \Phi(E_{ij}, E_{st}),$$

where the 2-cocycle Φ is defined by:

$$\Phi(A, B) = \text{tr}([J, A]B), \quad J = \sum_{i \leq 0} E_{ii}.$$

Recall that $\widehat{gl} = \widetilde{gl} + \mathbb{C}K$ is a central extension defined by the cocycle Φ of the Lie algebra \widetilde{gl} of all matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ with finitely many non-zero diagonals. We have also the following commutation relations:

$$(2.8) \quad [E_{ij}, \gamma_{-m}^k] = \delta_{jm} \gamma_{-i}^k, \quad [E_{ij}, \gamma_m^{*k}] = -\delta_{im} \gamma_j^{*k}.$$

These formulas define a representation of \widehat{gl} on W_N by derivations.

Introduce the following subspaces of the algebra W_N :

$$\begin{aligned} U_m &= \sum_{i=1}^N \mathbb{C} \gamma_{-m}^i, & U_m^* &= \sum_{i=1}^N \mathbb{C} \gamma_m^{*i}, \\ U &= \sum_{m \in \mathbb{Z}} U_m, & U^* &= \sum_{m \in \mathbb{Z}} U_m^*, \\ U_- &= \sum_{m \leq 0} U_m, & U_-^* &= \sum_{m \leq 0} U_m^*, \\ U_+ &= \sum_{m > 0} U_m, & U_+^* &= \sum_{m > 0} U_m^*. \end{aligned}$$

The following observations are clear by (2.5).

Remark 2.1. The subspaces U_m (resp. U_m^*) are gl_N -submodules of W_N isomorphic to the standard gl_N -module \mathbb{C}^N (resp. its dual).

Remark 2.2. As a gl_N -module (with respect to the adjoint representation), W_N is isomorphic to the symmetric algebra over the gl_N -module $U + U^*$.

Since $gl_N|0\rangle = 0$, we obtain

Remark 2.3. As a gl_N -module, M is isomorphic to the symmetric algebra over the gl_N -module $U_-^* + U_+$.

Proposition 2.1. *The centralizer $(W_N)^{gl_N}$ of the action of gl_N on the algebra W_N is an associative subalgebra generated by the elements*

$$E_{ij} = \sum_{p=1}^N : \gamma_{-i}^p \gamma_j^{*p} :,$$

where $i, j \in \mathbb{Z}$.

Proof. Due to Remarks 2.1 and 2.2, the proposition follows from the first fundamental theorem of classical invariant theory for GL_N (which states that the algebra

of invariant polynomials on the direct sum of any number of copies of the standard GL_N -module \mathbb{C}^N and its dual is generated by the obvious invariant polynomials of degree two, see *e.g.* [VP]). \square

Due to Remarks 2.1 and 2.3, M decomposes in a direct sum of finite-dimensional irreducible gl_N -modules. Let

$$(2.9) \quad M = \bigoplus_E M_E$$

be the decomposition of M in a direct sum of isotypic components with respect to gl_N . It is easy to prove now the first main result of the paper.

Theorem 2.1. *Each isotypic component M_E in (2.9) is irreducible with respect to the Lie algebra $gl_N \oplus \widehat{gl}$.*

Proof. Due to Remarks 2.1–2.3, all conditions of Theorem 1.1 hold for the (gl_N, W_N) -module M . It follows that each M_E is an irreducible $(gl_N, W_N^{gl_N})$ -module. Now the theorem follows from Proposition 2.1. \square

3. THE DECOMPOSITION OF M WITH RESPECT TO $gl_N \oplus \widehat{gl}$

Let

$$\begin{aligned} gl_- &= \sum_{i,j \leq 0} \mathbb{C}E_{ij} \subset W_N^-, \\ gl_+ &= \sum_{i,j > 0} \mathbb{C}E_{ij} \subset W_N^+. \end{aligned}$$

These are Lie subalgebras of the algebras W_N^\pm viewed as Lie algebras with the usual bracket. They are also subalgebras of the Lie algebra \widehat{gl} (the restriction of the 2-cocycle Φ to these subalgebras is trivial). Of course gl_- (resp. gl_+) is naturally identified with the Lie algebra of all matrices (a_{ij}) with only finite number of non-zero entries where $i, j \leq 0$ (resp. > 0).

Let \mathfrak{b}_N (resp. \mathfrak{b}_\pm) be the Borel subalgebra of all upper triangular matrices in gl_N (resp. gl_\pm). Let $I_N = \{1, \dots, N\}$, $I_+ = \{1, 2, 3, \dots\}$, $I_- = \{\dots, -2, -1, 0\}$. Given $\lambda = \{\lambda_i\}_{i \in I_N \text{ (resp. } I_\pm)}$ there exists a unique irreducible module $L_N(\lambda)$ (resp. $L_\pm(\lambda)$) which admits a (unique up to a constant factor) non-zero vector v_λ such that $\mathbb{C}v_\lambda$ is invariant with respect to the Borel subalgebra \mathfrak{b}_N (resp. \mathfrak{b}_\pm) and

$$E_{ii}v_\lambda = \lambda_i v_\lambda, \quad i \in I_N \text{ (resp. } I_\pm).$$

These modules are called irreducible highest weight modules (with highest weight λ).

For example the standard gl_N -module \mathbb{C}^N (resp. its dual) is the irreducible module with highest weight $(1, 0, \dots, 0)$ (resp. $(0, \dots, 0, -1)$). Tensor products of copies of the standard gl_N -module (resp. its dual) decompose into a direct sum of finite-dimensional irreducible modules with highest weights from the set

$$H_N^+ \text{ (resp. } H_N^-) = \{(\lambda_i)_{i \in I_N} \mid \lambda_i \text{ (resp. } -\lambda_i) \in \mathbb{Z}_+, \lambda_i \geq \lambda_j \text{ if } i < j\}.$$

The standard gl_+ (resp. gl_-)-module is $\mathbb{C}^{+\infty} = \bigoplus_{j > 0} \mathbb{C}v_j$ (resp. $\mathbb{C}^{-\infty} = \bigoplus_{j \leq 0} \mathbb{C}v_j$) with the usual action $E_{ij}v_k = \delta_{jk}v_i$. The gl_+ -module $\mathbb{C}^{+\infty}$ (resp. gl_- -module $\mathbb{C}^{-\infty}$) is an irreducible module with highest weight $(1, 0, 0, \dots)$ (resp.

$(\dots, 0, 0, -1)$). Tensor products of copies of the standard gl_+ (resp. gl_-) decompose into a direct sum of irreducible modules with highest weights from the set

$$H_{\pm} = \{(\lambda_i)_{i \in I_{\pm}} \mid \pm \lambda_i \in \mathbb{Z}_+, \lambda_i \geq \lambda_j \text{ if } i < j\}.$$

We shall identify H_N^{\pm} with a subset of H_{\pm} via the following map:

$$(\lambda_1, \dots, \lambda_N) \mapsto \begin{cases} (\lambda_1, \dots, \lambda_N, 0, 0, \dots) & \text{in } + \text{ case,} \\ (\dots, 0, 0, \lambda_1, \dots, \lambda_N) & \text{in } - \text{ case.} \end{cases}$$

For convenience of notation we let $\mathbb{C}^{-N} = (\mathbb{C}^N)^*$.

Lemma 3.1. *The $gl_N \oplus gl_{\pm}$ -module $S(\mathbb{C}^{\pm N} \otimes \mathbb{C}^{\pm \infty})$ has the following decomposition into a direct sum of irreducible modules:*

$$S(\mathbb{C}^{\pm N} \otimes \mathbb{C}^{\pm \infty}) = \bigoplus_{\lambda \in H_N^{\pm}} L_N(\lambda) \otimes L_{\pm}(\lambda).$$

Proof. The proof follows from the well-known Cauchy's formula (see e.g. [M]):

$$\frac{1}{\prod_{i=1}^N \prod_{j=1}^{\infty} (1 - x_i y_j)} = \sum_{\lambda \in H_N^+} \text{ch } L_N(\lambda) \text{ch } L_+(\lambda),$$

where $\text{ch } L_N(\lambda) = \text{tr}_{L_N(\lambda)} \text{diag}(x_1, \dots, x_N)$ and $\text{ch } L_+(\lambda) = \text{tr}_{L_+(\lambda)} \text{diag}(y_1, y_2, \dots)$ are characters. \square

Let

$$\mathfrak{g} = gl_N \oplus gl_- \oplus gl_+$$

be a direct sum of Lie algebras. Commutation relations (2.5) and (2.8) imply the following

Remark 3.1. We have:

$$\begin{aligned} [gl_-, U_-^*] &\subset U_-^*, [gl_-, U_+] = 0, \\ [gl_+, U_+] &\subset U_+, [gl_+, U_-^*] = 0 \end{aligned}$$

Thus, both U_-^* and U_+ are \mathfrak{g} -modules. The \mathfrak{g} -module U_+ (resp. U_-^*) is isomorphic to the $gl_N \oplus gl_+$ (resp. $gl_N \oplus gl_-$)-module $\mathbb{C}^N \otimes \mathbb{C}^{+\infty}$ (resp. $\mathbb{C}^{N*} \otimes \mathbb{C}^{-\infty}$) with the trivial action of gl_- (resp. gl_+).

Since $gl_{\pm}|0\rangle = 0$, we obtain, using also Remark 2.3:

Remark 3.2. As a \mathfrak{g} -module, M is isomorphic to the symmetric algebra over $U_-^* \oplus U_+$.

It is easy now to decompose the metaplectic representation M with respect to \mathfrak{g} . First, note that, by Remark 3.2 we have the following isomorphism of \mathfrak{g} -modules

$$(3.1) \quad M \simeq S(U_-^*) \otimes S(U_+).$$

The decomposition of each of the factors is given by Lemma 3.1, using Remark 3.1. In order to state the result, introduce the following notation. Given finite-dimensional irreducible gl_N -modules $L_N(\lambda)$ and $L_N(\mu)$, write their tensor product decomposition:

$$(3.2) \quad L_N(\lambda) \otimes L_N(\mu) = \bigoplus_{\nu} c_{\lambda,\mu}^{\nu} L_N(\nu), \quad c_{\lambda,\mu}^{\nu} \in \mathbb{Z}_+.$$

Note that if $\lambda \in H_N^-$, $\mu \in H_N^+$, then $\nu \in H_N$, where

$$H_N = \{(\lambda_i)_{i \in I_N} \mid \lambda_i \in \mathbb{Z}, \quad \lambda_i \geq \lambda_j \text{ if } i < j\}.$$

Thus, we arrive at the following result.

Proposition 3.1. *The following is a decomposition of M as a \mathfrak{g} -module:*

$$M = \bigoplus_{\substack{\lambda \in H_N^- \\ \mu \in H_N^+ \\ \nu \in H_N}} c_{\lambda,\mu}^{\nu} L_N(\nu) \otimes L_-(\lambda) \otimes L_+(\mu). \quad \square$$

Choose in each gl_N -module $L_N(\nu)$, $\nu \in H_N$, the 1-dimensional subspace $\mathbb{C}v_{\nu}$. Then, the subspace

$$M^{\nu} := \{m \in M \mid \mathfrak{b}_N m \subset \mathbb{C}m, \quad e_{ii}m = \nu_i m\}$$

is a \widehat{gl} -submodule, which is irreducible due to Theorem 2.1. The decomposition (1.1) becomes:

$$(3.3) \quad M = \bigoplus_{\nu \in H_N} L_N(\nu) \otimes M^{\nu}$$

as $gl_N \oplus \widehat{gl}$ -modules. Due to Proposition 3.1, as a $gl_- \oplus gl_+$ -module, M^{ν} ($\nu \in H_N$) decomposes as follows:

$$(3.4) \quad M^{\nu} = \bigoplus_{\substack{\lambda \in H_N^- \\ \mu \in H_N^+}} c_{\lambda,\mu}^{\nu} L_-(\lambda) \otimes L_+(\mu).$$

Denote by \mathfrak{b} the subalgebra of \widehat{gl} of upper triangular matrices plus $\mathbb{C}K$, where K is a central element acting on M (and each M^{ν}) as $-NI$, and let \mathfrak{n} be the subalgebra of \mathfrak{b} of matrices with 0's on the diagonal. It is clear that \mathfrak{n} acts locally nilpotently on M . Hence each \widehat{gl} -module M^{ν} is an irreducible highest weight module $L(\Lambda(\nu), -N)$. Recall that the irreducible highest weight \widehat{gl} -module $L(\Lambda, c)$, where $\Lambda = (\Lambda_j)_{j \in \mathbb{Z}}$ and $c \in \mathbb{C}$, is defined by the properties that there exists a unique up to a constant multiple vector $v \in L(\Lambda, c)$ such that $\mathfrak{n}(v) = 0$ and

$$E_{ii}v = \Lambda_i v, \quad i \in \mathbb{Z}; \quad K = cI.$$

We let $\Lambda(\nu) = (\Lambda(\nu)_j)_{j \in \mathbb{Z}}$. It remains to calculate $\Lambda(\nu)$ for each $\nu \in H_N$.

We claim that

$$(3.5) \quad \Lambda(\nu) = (\dots, 0, \nu_{p+1}, \dots, \nu_N; \nu_1, \nu_2, \dots, \nu_p, 0, \dots),$$

where $p = 0$ if all $\nu_i < 0$, $p = N$ if all $\nu_i > 0$, and $1 \leq p < N$ is such that $\nu_p \geq 0 \geq \nu_{p+1}$ otherwise. We put here semicolon between the 0th and the first slots (and comma in all other places).

The proof of (3.5) will follow from a sequence of combinatorial lemmas.

Let us define S to be the set of pairs of N -tuples $(\lambda; \mu)$ such that $\lambda_i, \mu_j \in \mathbb{Z}$, where $\lambda_i \leq 0, \mu_j \geq 0$ for all i, j .

Let us define operations $R_{ij}^a : (\lambda; \mu) \rightarrow (\lambda'; \mu')$, $a = 1, 2, 3$, $1 \leq i, j \leq N$ provided that $(\lambda'; \mu') \in S$:

$$\begin{aligned} R_{ij}^1 : \quad & \lambda'_i = \lambda_i - 1, \quad \lambda'_j = \lambda_j + 1 \quad \text{for } i > j, \\ R_{ij}^2 : \quad & \mu'_i = \mu_i - 1, \quad \mu'_j = \mu_j + 1 \quad \text{for } i > j, \\ R_{ij}^3 : \quad & \lambda'_i = \lambda_i + 1, \quad \mu'_j = \mu_j - 1 \quad \text{for any } i, j, \end{aligned}$$

all other entries of $(\lambda; \mu)$ do not change.

We will say that $(\lambda''; \mu'') \geq (\lambda; \mu)$ if one may obtain $(\lambda''; \mu'')$ from $(\lambda; \mu)$ by a sequence of operations R_{ij}^a .

We define a subset S_ν in S by two conditions: (a) λ, μ are dominant gl_N weights, (b) $L_N(\lambda) \otimes L_N(\mu) \supset L_N(\nu)$.

Remark 3.3. A map from S into \widehat{gl} -weights given by

$$(\lambda; \mu) \mapsto (\dots, 0, 0, \lambda_1, \dots, \lambda_N; \mu_1, \dots, \mu_N, 0, 0, \dots)$$

sends S_ν into set of $gl_- \oplus gl_+$ -highest weights of the \widehat{gl} -module $L(\Lambda(\nu))$ and the ordering on S into the standard ordering of \widehat{gl} -weights.

We will describe condition (b) in the definition of S_ν by the table

$\mu_1 \dots \mu_p$	$\mu_{p+1} \dots \mu_N$
$\lambda_1 \dots \lambda_p$	$\lambda_{p+1} \dots \lambda_N$
$\nu_1 \dots \nu_p$	$\nu_{p+1} \dots \nu_N$
$q_1 \dots q_p$	$q_{p+1} \dots q_N$

Here (q_1, \dots, q_N) is a sum of negative gl_N roots such that $\mu_i + \lambda_i + q_i = \nu_i$.

Let $Q^i = \sum_{j \leq i} q_j$.

Lemma 3.2. (q_1, \dots, q_N) is a sum of negative roots if and only if

$$(3.6) \quad Q^i \leq 0; \quad Q^N = 0$$

Proof. Let q_a be the negative number with maximal a , q_b the positive number with minimal $b > a$. If we add to (q_1, \dots, q_N) the (positive) root of $E_{a,b}$ the condition (3.6) remains valid and $\sum_i |q_i|$ decreases by 2.

Lemma 3.3. *Let $(\lambda; \mu) \in S_\nu$. Then*

- (a) $\mu_i + q_i \geq 0$,
- (b) $\lambda_i + q_i \leq 0$,
- (c) $\mu_j \geq \nu_j$,
- (d) $\lambda_i \leq \nu_i$.

Proof. Since $\lambda_i + \mu_i + q_i = \nu_i$, then (a) implies (d) and (b) implies (c). To prove (a) we will use the description of the decomposition of the tensor product of two gl_N modules with dominant highest weights based on Weyl determinants (see [Z]). In particular this description implies the following statement. If $L_N(\lambda) \otimes L_N(\mu) \supset L_N(\nu)$ and $\mu_i \geq 0$ for all i , then one can find the weight (ν_1, \dots, ν_N) among the N -tuples obtained from $(\lambda_1, \dots, \lambda_N)$ by a sequence of operations ($m \in \mathbb{Z}_+$):

$$\Gamma_m(\lambda_1, \dots, \lambda_N) = \sum'_{\substack{\alpha_k \geq 0 \\ \alpha_1 + \dots + \alpha_N = m}} (\lambda_1 + \alpha_1, \dots, \lambda_N + \alpha_N)$$

where one drops $(\lambda_1 + \alpha_1, \dots, \lambda_N + \alpha_N)$ from the sum if at least one of the weights $(\lambda_1, \dots, \lambda_i + \alpha_i, \dots, \lambda_N)$ is not dominant. Hence $\mu_i + q_i \geq 0$. Since $L_N(\lambda)^* \otimes L_N(\lambda)^* \supset L_N(\nu)^*$ the same reasoning proves (b).

Lemma 3.4. *Let $r \in [0, p]$, $s \in [p+1, N]$. Applying a sequence of operations $R_{s,r}^1$ and $R_{s,r}^2$ one may get from $(\lambda; \mu) \in S_\nu$ a sequence $(\lambda'', \mu'') \in S$ such that*

- (a) $\mu_s'' = 0$; $\lambda_r'' = 0$,
- (b) $\mu_s'' + q_s'' \geq 0$; $\lambda_r'' + q_r'' \leq 0$, where (q_1'', \dots, q_N'') is a sum of negative roots,
- (c) $q_r'' \leq 0$; $q_s'' \geq 0$.

Proof. It is clear that (a) and (b) imply (c). By Lemma 3.3, (c) is valid. Let $\mu_s > 0$ for some s . Let us apply $R_{s,r}^2$ for some r . Then $\mu_s' = \mu_s - 1$, $\mu_r' = \mu_r + 1$; $q_s' = q_s + 1$, $q_r' = q_r - 1$, hence we add a negative root to (q_1, \dots, q_N) . Moreover, $\mu_i + q_i$ do not change for all i ; $\lambda_r' + q_r' = \lambda_r + q_r - 1$, hence $\lambda_r' + q_r'$ is still nonpositive for all r . Similar considerations apply to $R_{s,r}^1$. \square

Lemma 3.5. *Let a pair $(\lambda; \mu) \in S$ be described by a table*

$\mu_1 \cdots \mu_p$	$0 \cdots 0$
$0 \cdots 0$	$\lambda_{p+1} \cdots \lambda_N$
$\nu_1 \cdots \nu_p$	$\nu_{p+1} \cdots \nu_N$
$q_1 \cdots q_p$	$q_{p+1} \cdots q_N$

such that

- (a) $\nu_1 \geq \dots \geq \nu_p \geq 0 \geq \nu_{p+1} \geq \dots \geq \nu_N$, $\mu_i + \lambda_i + q_i = \nu_i$ where (q_1, \dots, q_N) is a sum of negative roots;
- (b) $q_r \leq 0$, $q_s \geq 0$.

Then applying a sequence of operations $R_{r,s}^3$ one may get a pair (λ'', μ'') with $q_i'' = 0$.

Proof. If $q_r < 0$ for some r , then one may find s such that $q_s > 0$. Hence $\mu_r > 0$ and $\lambda_s < 0$, and one may apply $R_{s,r}^3$. It is clear that $Q^i < 0$ for $i < s$, hence $\{q_j'\}$ is still a sum of negative roots, and $\sum_i |q_i|$ decreases by 2. \square

Proof of Theorem 3.1. It is clear that the sequence on the right of (3.5) lies on S_ν . Lemmas 3.3–3.5 and Remark 3.3 imply that the corresponding \widehat{gl} -weight is maximal. This completes the proof of (3.5). \square

Thus we proved our next main result.

Theorem 3.1. *With respect to $gl_N \oplus \widehat{gl}$ the metaplectic representation M decomposes as follows:*

$$(3.7) \quad M = \bigoplus_{\nu \in H_N} L_N(\nu) \otimes L(\Lambda(\nu), -N)$$

where $\Lambda(\nu)$ is defined by (3.5).

For us the most important consequence of this result is

Corollary 3.1. $M^{gl_N} \simeq L(0, -N)$ as \widehat{gl} -modules.

As another corollary of (3.7) and (3.4), we obtain a character formula for the irreducible \widehat{gl} -module $L(\Lambda(\nu), -N)$ with central charge $-N$ and highest weight of the form $\Lambda(\nu)$ given by (3.5) where $\nu \in H_N$:

$$(3.8) \quad \text{tr}_{L(\Lambda(\nu), -N)} \text{diag}(\dots, y_2^{-1}, y_1^{-1}; x_1, x_2, \dots) = \sum_{\lambda, \mu \in H_N^+} c_{\lambda^* \mu}^\nu S_\lambda(y) S_\mu(x).$$

Here for $\lambda = (\lambda_1, \dots, \lambda_N) \in H_N^+$ we let $\lambda^* = (-\lambda_N, \dots, -\lambda_1) (\in H_N^-)$, $S_\mu(x)$ stands for $\text{tr}_{L_+(\mu)} \text{diag}(x_1, x_2, \dots)$, and $c_{\lambda^* \mu}^\nu$ are defined by (3.2). Formula (3.8) follows from (3.4) and the proof of Theorem 3.1. In particular, for the *vacuum* \widehat{gl} -module $L(0, -N)$ with central charge $-N$ we obtain:

$$(3.9) \quad \text{tr}_{L(0, -N)} \text{diag}(\dots, y_2^{-1}, y_1^{-1}; x_1, x_2, \dots) = \sum_{\lambda \in H_N^+} S_\lambda(y) S_\lambda(x).$$

This last formula was stated in [AFMO].

4. ON COMPLETE REDUCIBILITY OF CERTAIN \widehat{gl} -MODULES

Recall that labels of the highest weight $\Lambda = (\Lambda_j)_{j \in \mathbb{Z}}$ of the \widehat{gl} -module $L(\Lambda, c)$ are the numbers $n_i = \Lambda_i - \Lambda_{i+1} + \delta_{i,0}c$ ($i \in \mathbb{Z}$). It is clear that a sequence $\{n_i\}_{i \in \mathbb{Z}}$ is the sequence of labels of a highest weight Λ of a \widehat{gl} -module $L(\Lambda, -N)$ that occurs in the decomposition (3.7) iff the following properties hold:

$$(4.1a) \quad n_i \in \mathbb{Z}_+ \quad \text{if} \quad i \neq 0 \quad \text{and} \quad \sum_i n_i = -N,$$

$$(4.1b) \quad \text{if } n_i \neq 0 \quad \text{and} \quad n_j \neq 0, \quad \text{then} \quad |i - j| \leq N.$$

We denote, as usual, by O_{-N} the category of \widehat{gl} -modules for which the subalgebra \mathfrak{b} acts locally finitely and $c = -N$. All irreducible subquotients of a module from the category O_{-N} are the modules $L(\Lambda, -N)$. The following is the main result of this section.

Theorem 4.1. *If all irreducible subquotients of a \widehat{gl} -module from the category O_{-N} have labels satisfying conditions (4.1a,b), then this module is a direct sum of irreducible \widehat{gl} -modules.*

First, we translate the problem to that for the Lie algebra gl_{fin} of matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ with only a finite number of non-zero a_{ij} 's. We denote by \mathfrak{b}_{fin} the subalgebra of upper triangular matrices and by $L(\Lambda)$ the irreducible gl_{fin} -module defined

by the property that there exists an eigenvector v_Λ for \mathfrak{b} such that $E_{ii}v_\Lambda = \Lambda_i v_\Lambda$; we let $\Lambda = (\Lambda_i)_{i \in \mathbb{Z}}$.

Consider the homomorphism $\varphi : gl_{fin} \rightarrow \widehat{gl}$ defined by:

$$\varphi(E_{ij}) = E_{ij} \text{ if } i \neq j \text{ or } i = j \leq 0, \quad \varphi(E_{ii}) = E_{ii} - K \text{ if } i > 0.$$

Define a sequence $\tilde{N} = (\tilde{N}_i)_{i \in \mathbb{Z}}$ by

$$\tilde{N}_i = 0 \text{ if } i \leq 0, \quad \tilde{N}_i = N \text{ if } i > 0.$$

Then the set of highest weights that occurs in the decomposition (3.7), when restricted to gl_{fin} via φ , is characterised by the following properties:

$$(4.2a) \quad \Lambda_i \in \mathbb{Z}, \quad \Lambda_i = 0 \text{ for } i \leq -N - 1 \text{ and } \Lambda_i = N \text{ for } i \geq N,$$

$$(4.2b) \quad \Lambda_i \geq \Lambda_{i+1} \text{ if } i \neq 0,$$

$$(4.2c) \quad \text{if } \Lambda_i \neq \tilde{N}_i \text{ and } \Lambda_j \neq \tilde{N}_j, \text{ then } |i - j| \leq N - 1.$$

It is clear that Theorem 4.1 follows from the analogous statement for the Lie algebra gl_{fin} :

Proposition 4.1. *If \mathfrak{b}_{fin} acts locally finitely in a gl_{fin} -module and all the irreducible subquotients of this module have highest weights satisfying conditions (4.2a–c), then this module is a direct sum of irreducible gl_{fin} -modules.*

Let S_∞ be the group of all permutations σ of \mathbb{Z} such that $\sigma(i) = i$ for all but finitely many i . Define the weight ρ by $\rho_i = -i$ ($i \in \mathbb{Z}$). Since gl_{fin} is the inductive limit of the Lie algebras gl_n , by highest weight representation theory for gl_n (see e.g. [D]) Proposition 4.1 follows from

Proposition 4.2. *Let both $\Lambda = (\Lambda_i)_{i \in \mathbb{Z}}$ and $M = (M_i)_{i \in \mathbb{Z}}$ satisfy conditions (4.2a–c) and let*

$$(4.3) \quad \sigma(\Lambda + \rho) = M + \rho \text{ for some } \sigma \in S_\infty.$$

Then $\Lambda = M$.

Proof. Suppose that $j < -N$ is not fixed by σ and let $j' = \sigma(j)$. Then we have:

$$(\Lambda + \rho)_j = (M + \rho)_j = -j \text{ and } (M + \rho)_{j'} = -j.$$

It follows that $j' \in [-N, N - 1]$ and that the transposition (jj') of j and j' does not change $M + \rho$. Hence the permutation $\sigma' = (jj')\sigma$ still satisfies (4.3) and has fewer non-fixed points outside $[-N, N - 1]$. We argue similarly in the case $j \geq N$. Thus we may assume that

$$(4.4) \quad \sigma(j) = j \text{ for } j \notin [-N, N - 1].$$

Given $a \in \mathbb{Z}$ let

$$\mathcal{J}_a = \{i \in \mathbb{Z} | a < i \leq N\} \cup \{i \in \mathbb{Z} | -N - 1 < i \leq a - N\}.$$

By (4.2c) there exists $b \in [-N, N-1]$ such that

$$\Lambda_i = \tilde{N}_i \text{ for } i > b \text{ and for } i \leq b - N.$$

Then the set $\{(\Lambda + \rho)_i | i \in \mathcal{J}_b\}$ is a permutation of the set $[1, N]$ since $(\Lambda + \rho)_i = N - i$ for $i > b$ and $(\Lambda + \rho)_i = -i$ for $i \leq b - N$; we denote the corresponding bijective map of this set to the set of integers $[1, N]$ by τ_b . Similarly choose $b_1 \in [-N, N-1]$ such that

$$M_i = \tilde{N}_i \text{ for } i > b_1 \text{ and for } i \leq b_1 - N,$$

and define the map $\tau_{b_1} : \{(M + \rho)_i | i \in \mathcal{J}_b\} \rightarrow [1, N]$.

We may assume that in addition to properties (4.3) and (4.4), σ has the property

$$(4.5) \quad \sigma(i) = \tau_{b_1}^{-1} \tau_b(i) \text{ for } i \in \mathcal{J}_b.$$

Indeed, if (4.5) does not hold for some $i \in \mathcal{J}_b$, it means that $\sigma(i) \notin \mathcal{J}_{b_1}$. Then we have:

$$(\Lambda + \rho)_i = \tau_b(i), \quad (M + \rho)_{\tau_{b_1}^{-1} \tau_b(i)} = \tau_b(i) = (M + \rho)_{\sigma(i)}.$$

Hence the permutation $\sigma' = (\sigma(i), \tau_{b_1}^{-1} \tau_b(i))\sigma$ still satisfies (4.3) and (4.4) and reduces the number of $i \in \mathcal{J}_b$ which do not satisfy (4.5).

In order to complete the proof of Proposition 4.2 we need

Lemma 4.1. *Suppose that Λ , M , σ , b and b_1 satisfy (4.2)–(4.5), and assume that $b \geq b_1$. Then*

- (a) $\sigma(i) = i$ for $b - N < i \leq b_1$
- (b) $(\Lambda + \rho)_i = N - i$ for $b_1 < i \leq b$, $(M + \rho)_i = -i$ for $b_1 - N < i \leq b - N$.

Proof. Consider $b_1 + 1$ numbers $(\Lambda + \rho)_i$ for $0 \leq i \leq b_1$. This is a strictly decreasing sequence of integers and $(\Lambda + \rho)_{b_1} \geq N - b_1$. None of these integers is mapped by σ to $(M + \rho)_j$ for $b_1 - N < j < 0$ since $(M + \rho)_j < N - b_1$ for these j . Hence the strictly decreasing sequence $\{(\Lambda + \rho)_i\}_{0 \leq i \leq b_1}$ is mapped by σ to strictly decreasing sequence $\{(M + \rho)_j\}_{0 \leq j \leq b}$. So the only possibility for σ is:

$$\sigma(i) = i \text{ and } (\Lambda + \rho)_i = (M + \rho)_i \text{ for } 0 \leq i \leq b_1.$$

A similar argument works for $b - N < i < 0$, proving (a).

The proof of (b) is similar. The strictly decreasing sequence $\{(\Lambda + \rho)_i\}_{b_1 < i \leq b}$ is mapped by σ to strictly decreasing sequence $\{(M + \rho)_j\}_{b_1 - N < j \leq b - N}$. Since $(\Lambda + \rho)_b \geq N - b$ and $(M + \rho)_{b_1 - N + 1} \leq N - b_1 - 1$, the possibility described by (b) is the only one. \square

End of the proof of Proposition 4.2. Exchanging Λ and M and replacing σ by σ^{-1} , if necessary, we may assume that $b \geq b_1$. By (4.4) and Lemma 4.1b, $\Lambda_i = M_i$ for $i \notin (b - N, b_1]$; by Lemma 4.1a, $\Lambda_i = M_i$ for $i \in (b - N, b_1]$. \square

Remark 4.1. Consider the antilinear anti-involution of the Weyl algebra W_N defined by

$$(\gamma_m^i)^\dagger = \gamma_{-m}^{*i} \quad \text{if } m < 0, \quad (\gamma_m^i)^\dagger = -\gamma_{-m}^{*i} \quad \text{if } m \geq 0.$$

The unique Hermitean form on M , normalized by the condition that the length of the vacuum vector is 1 and such that the operator adjoint to $a \in W_N$ is a^\dagger , is positive definite. The anti-involution \dagger induces the compact anti-involution on gl_N and the following antilinear anti-involution on \widehat{gl} :

$$E_{ij}^\dagger = E_{ji} \quad \text{if } i, j > 0 \quad \text{or } i, j \leq 0, \quad E_{ij}^\dagger = -E_{ji} \quad \text{otherwise,}$$

so that on $gl_- \oplus gl_+$ it induces the compact involution. Thus all the \widehat{gl} -modules $L(\Lambda, -N)$ with Λ satisfying (4.1) are unitarizable with respect to the (non-compact) anti-involution \dagger . One may view (3.4) as their “ K -type decomposition.” Note also that, in view of [KV] and [O], these are all unitarizable highest weight \widehat{gl} -modules with central charge $-N$.

5. A DIGRESSION ON VERTEX ALGEBRAS AND THEIR TWISTED MODULES

We explain here the basics of the theory of vertex algebras (sometimes also called vertex operator algebras) that will be used in the sequel. Our definition of a vertex algebra V is close to that used in [FKRW] (we do not assume here that V is \mathbb{Z}_+ -graded) and one can show (cf. [G] and [K2]) that it is equivalent to the original definition of Borcherds [B1], [B2]. The reader may also consult [FLM] for a different formalism.

Let V be a vector space. A *field* is a series of the form

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

where $a_{(n)} \in \text{End } V$ are such that for each $v \in V$ one has: $a_{(n)}v = 0$ for $n \gg 0$. Here z is a formal indeterminate. We shall often use a different indexing of the modes of $a(z)$; in such a case the parenthesis around indices will be dropped, like $a(z) = \sum_n a_n z^{-n-\Delta}$.

A vertex algebra structure on V is a vector $|0\rangle \in V$ (called the vacuum vector) and a linear map of V to the space of fields $a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ (called the state-field correspondence), satisfying the following axioms:

- (T) $[T, Y(a, z)] = \partial_z Y(a, z)$, where $T \in \text{End } V$ is defined by $T(a) = a_{(-2)}|0\rangle$,
- (V) $Y(|0\rangle, z) = I_V$, $Y(a, z)|0\rangle|_{z=0} = a$,
- (L) $(z-w)^N [Y(a, z), Y(b, w)] = 0$ for $N \gg 0$.

Here and further the equality means a coefficient-wise equality of series in z or in z and w . These axioms are usually called the translation covariance, the vacuum and the locality axioms.

Let Γ be an additive subgroup of \mathbb{C} containing \mathbb{Z} . A Γ -twisting of V is a Γ/\mathbb{Z} -gradation $V = \bigoplus_{\bar{\alpha} \in \Gamma/\mathbb{Z}} \bar{\alpha}V$ such that

$$a_{(n)}^{\bar{\beta}} V \subset \bar{\alpha} + \bar{\beta} V \quad \text{if } a \in \bar{\alpha} V.$$

Here and further $\bar{\alpha}$ stands for the coset $\alpha + \mathbb{Z}$.

Let M be a vector space and let $\bar{\alpha} \in \Gamma/\mathbb{Z}$. An $\bar{\alpha}$ -twisted $\text{End } M$ -valued field is a series of the form $\sum_{n \in \bar{\alpha}} a_{(n)}^M z^{-n-1}$ where $a_{(n)}^M \in \text{End } M$ is such that $a_{(n)}^M v = 0$ for $v \in M$, $n \gg 0$. A Γ -twisted V -module M is a linear map from V to the linear

span of Γ -twisted $\text{End } M$ -valued fields, associating to each $a \in {}^{\bar{\alpha}}V$ an $\bar{\alpha}$ -twisted field $Y^M(a, z) = \sum_{n \in \bar{\alpha}} a_{(n)}^M z^{-n-1}$ such that the following three axioms hold:

- (M1) $Y^M(1, z) = I_M$,
- (M2) $Y^M(Ta, z) = \partial_z Y^M(a, z)$,
- (M3) (twisted Borcherds identity) $\sum_{j=0}^{\infty} \binom{m}{j} Y^M(a_{(n+j)}b, z)z^{m-j}$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left(a_{(m+n-j)}^M Y^M(b, z)z^j - (-1)^n Y^M(b, z) a_{(m+j)}^M z^{n-j} \right)$$

for all $m \in \bar{\alpha}$, $n \in \mathbb{Z}$.

In the case when $\Gamma = \mathbb{Z}$, M is called a (untwisted) V -module. Note that V itself is a V -module since one can show that properties (M2) and (M3) with $\bar{\alpha} = \mathbb{Z}$ and superscript M removed follow from the axioms (T), (V), and (L) of a vertex algebra (see [K2]).

Letting $n = 0$ in (M3) and substituting $Y^M(b, w) = \sum_{k \in \bar{\beta}} b_{(k)} z^{-k-1}$, we get

$$(5.1) \quad \left[a_{(m)}^M, b_{(n)}^M \right] = \sum_{j=0}^{\infty} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}^M, \quad m \in \bar{\alpha}, \quad n \in \bar{\beta}.$$

Letting $m = \alpha \in \bar{\alpha}$ and $n = -1$ in (M3) we get

$$(5.2) \quad \sum_{j=0}^{\infty} \binom{\alpha}{j} Y^M(a_{(j-1)}b, z)z^{-j} =: Y^M(a, z)Y^M(b, z) :,$$

where the normally ordered product is defined, as usual, by

$$: Y^M(a, z)Y^M(b, z) := Y^M(a, z)_+ Y^M(b, z) + Y^M(b, z)Y^M(a, z)_-$$

and

$$Y^M(a, z)_- = \sum_{j \in \alpha + \mathbb{Z}_+} a_{(j)}^M z^{-j-1}, \quad Y^M(a, z)_+ = Y^M(a, z) - Y^M(a, z)_-$$

are the *annihilation* and the *creation* parts of $Y^M(a, z)$. Letting $m = \alpha \in \bar{\alpha}$ and replacing n by $-n - 1$ with $n \in \mathbb{Z}_+$ in (M3) gives:

$$(5.3) \quad \sum_{j=0}^{\infty} \binom{\alpha}{j} Y^M(a_{(-n+j-1)}b, z)z^{-j} =: \partial^{(n)} Y^M(a, z)Y^M(b, z) :,$$

where $\partial^{(n)}$ stands for $\partial_z^n / n!$. It is easy to show that (5.1) and (5.2) along with (M2) imply (M3) (or (5.1) and (5.3) along with (M1) imply (M2) and (M3)).

It is well-known that the space M of the metaplectic representation carries a structure of a vertex algebra with vacuum vector $|0\rangle$ and with the state-field correspondence defined as follows ($m_i, n_i \in \mathbb{Z}_+$; $1 \leq i_1, \dots, i_s, j_1, \dots, j_t \leq N$):

$$\begin{aligned} Y \left(\gamma_{-m_1-1}^{i_1} \cdots \gamma_{-m_s-1}^{i_s} \gamma_{-n_1}^{*j_1} \cdots \gamma_{-n_t}^{*j_t} |0\rangle, z \right) \\ =: \partial^{(m_1)} \gamma^{i_1}(z) \cdots \partial^{(m_s)} \gamma^{i_s}(z) \partial^{(n_1)} \gamma^{*j_1}(z) \cdots \partial^{(n_t)} \gamma^{*j_t}(z) :. \end{aligned}$$

Here the normally ordered product of more than two fields is taken from right to left.

Fix a non-zero complex number s and let Γ_s denote the additive subgroup of \mathbb{C} generated by s and 1. Define a Γ_s -twisting of M by letting the twist of

$$\gamma_{m_1}^{i_1} \cdots \gamma_{m_u}^{i_u} \gamma_{n_1}^{*j_1} \cdots \gamma_{n_\ell}^{*j_\ell} |0\rangle \quad \text{equal} \quad s(u-t) + \mathbb{Z}.$$

We construct a Γ_s -twisted M -module M_s as follows (*cf.* [ABMNF]). As a vector space we take $M_s = M$ and let

$$\gamma_s^i(z) \equiv Y^{M_s}(\gamma_{-1}^i |0\rangle, z) := z^{-s} \gamma^i(z), \quad \gamma_s^{*i}(z) \equiv Y^{M_s}(\gamma_0^{*i} |0\rangle, z) := z^s \gamma^{*i}(z).$$

By making use of (M1) and (5.3), this extends inductively to a Γ_s -twisted module structure on M_s . We shall write $Y_s(a, z) = Y^{M_s}(a, z)$ to simplify the notation.

Denote by $W_{1+\infty, -N}$ the vertex subalgebra of the vertex algebra M generated by the fields

$$J^k(z) = \sum_{i=1}^N : \gamma^i(z) \partial_z^k \gamma^{*i}(z) :$$

(*i.e.*, $W_{1+\infty, -N}$ is the subspace of M consisting of polynomials of the modes of the $J^k(z)$ applied to $|0\rangle$). Note that we have in the domain $|\epsilon| < |z|$:

$$\sum_{k=0}^{\infty} J^k(z) \epsilon^k / k! = E(z, z + \epsilon).$$

Due to Corollary 3.1, it follows that

$$(5.4) \quad W_{1+\infty, -N} = M^{gl_N}.$$

Note that for all Γ_s -twistings of M we have

$$W_{1+\infty, -N} \subset {}^0M.$$

It follows that the restriction of the Γ_s -twisted M -module M_s to the vertex subalgebra $W_{1+\infty, -N}$ is a (untwisted) $W_{1+\infty, -N}$ -module. We have by (5.2) for $1 \leq i \leq N$, $k \in \mathbb{Z}_+$:

$$(5.5) \quad : Y_s(\gamma_{-1}^i |0\rangle, z) Y_s(\gamma_{-k}^{*i} |0\rangle, z) := Y_s(\gamma_{-1}^i \gamma_{-k}^{*i} |0\rangle, z) + \binom{s}{k+1} z^{-k-1} I.$$

Noting that

$$(5.6) \quad \frac{1}{k!} J^k(z) = Y \left(\sum_{i=1}^N \gamma_{-1}^i \gamma_{-k}^{*i} |0\rangle, z \right)$$

we obtain from (5.5) and (5.3)

$$(5.7) \quad \begin{aligned} J_s^k(z) &\equiv k! Y_s \left(\sum_{i=1}^N \gamma_{-1}^i \gamma_{-k}^{*i} |0\rangle, z \right) = \sum_{i=1}^N : \gamma_s^i(z) \partial_z^k \gamma_s^{*i}(z) : \\ &\quad + N \frac{s(s-1) \cdots (s-k)}{k+1} z^{-k-1} I. \end{aligned}$$

6. THE VERTEX ALGEBRA $W_{1+\infty,c}$ AND ITS MODULES AT $c = -N$

Let \mathcal{D} be the Lie algebra of complex regular differential operators on \mathbb{C}^\times with the usual bracket, in an indeterminate t . The elements

$$(6.1) \quad J_k^l = -t^{l+k}(\partial_t)^l \quad (k \in \mathbb{Z}, l \in \mathbb{Z}_+)$$

form a basis of \mathcal{D} . The Lie algebra \mathcal{D} has the following 2-cocycle with values in \mathbb{C} [KP, p. 3310]:

$$(6.2) \quad \Psi(f(t)\partial_t^m, g(t)\partial_t^n) = \frac{m!n!}{(m+n+1)!} \text{Res}_{t=0} f^{(n+1)}(t)g^{(m)}(t)dt,$$

where $f^{(m)}(t) = \partial_t^m f(t)$. We denote by $\widehat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}C$, where C is the central element, the corresponding central extension of the Lie algebra \mathcal{D} .

Another important basis of \mathcal{D} is

$$(6.3) \quad L_k^l = -t^k D^l \quad (k \in \mathbb{Z}, l \in \mathbb{Z}_+)$$

where $D = t\partial_t$. These two bases are related by the formula [KR]:

$$(6.4) \quad J_k^l = -t^k D(D-1) \cdots (D-l+1).$$

Given a sequence of complex numbers $\lambda = (\lambda_j)_{j \in \mathbb{Z}_+}$ and a complex number c there exists a unique irreducible $\widehat{\mathcal{D}}$ -module $L(\lambda, c; \widehat{\mathcal{D}})$ which admits a non-zero vector v_λ such that:

$$L_k^j v_\lambda = 0 \quad \text{for } k > 0, \quad L_0^j v_\lambda = \lambda_j v_\lambda, \quad C = cI.$$

This is called a highest weight module over $\widehat{\mathcal{D}}$ with highest weight λ and central charge c . The module $L(\lambda, c; \widehat{\mathcal{D}})$ is called *quasifinite* if all eigenspaces of D are finite-dimensional (note that D is diagonalizable). It was proved in [KR, Theorem 4.2] that $L(\lambda, c; \widehat{\mathcal{D}})$ is a quasi-finite module if and only if the generating series

$$\Delta_\lambda(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \lambda_n$$

has the form

$$(6.5) \quad \Delta_\lambda(x) = \frac{\phi(x)}{e^x - 1},$$

where

$$(6.6) \quad \phi(x) + c = \sum_i p_i(x) e^{r_i x} \quad (\text{a finite sum}),$$

$p_i(x)$ are non-zero polynomials in x such that $\sum_i p_i(0) = c$ and r_i are distinct complex numbers. The numbers r_i are called the *exponents* of this module and the polynomials $p_i(x)$ are called their *multiplicities*. One has the following nice property of quasi-finite $\widehat{\mathcal{D}}$ -modules:

Proposition 6.1. [KR, Theorem 4.8]. *We have:*

$$L(\lambda, c; \hat{\mathcal{D}}) \otimes L(\lambda', c; \hat{\mathcal{D}}) \simeq L(\lambda + \lambda', c + c'; \hat{\mathcal{D}})$$

provided that the difference of any exponent of the first module and any exponent of the second module is not an integer.

We call a quasifinite $\hat{\mathcal{D}}$ -module $L(\lambda, c; \hat{\mathcal{D}})$ *primitive* if all multiplicities of its exponents are integers. (Note that we required in [FKRW] the n_i to be positive, but we have to drop this condition here.)

Given $s \in \mathbb{C}$, we may consider the natural action on the algebra \mathcal{D} on the $t^s \mathbb{C}[t, t^{-1}]$. Choosing the basis $v_j = t^{-j+s}$ ($j \in \mathbb{Z}$) of this space gives us a homomorphism of Lie algebras $\phi_s : \mathcal{D} \rightarrow \tilde{gl}$:

$$(6.7) \quad \phi_s(t^k f(D)) = \sum_{j \in \mathbb{Z}} f(-j+s) E_{j-k, j}$$

The homomorphism ϕ_s lifts to the homomorphism $\hat{\phi}_s$ of the corresponding central extensions as follows [KR]:

$$\begin{aligned} \hat{\phi}_s(L_k^j) &= \phi_s(L_k^j) \text{ if } k \neq 0, \\ \hat{\phi}_s(e^{xD}) &= \phi_s(e^{xD}) - \frac{e^{sx} - 1}{e^x - 1} K, \quad \hat{\phi}_s(C) = K. \end{aligned}$$

It is straightforward to check using (6.4) that in the basis J_k^n this homomorphism becomes:

$$(6.8) \quad \begin{aligned} \hat{\phi}_s(J_k^n) &= \phi_s(J_k^n) \text{ if } k \neq 0, \\ \hat{\phi}_s(J_0^n) &= \phi_s(J_0^n) - \frac{s(s-1) \cdots (s-n)}{n+1} K, \quad \hat{\phi}_s(C) = K. \end{aligned}$$

The following proposition is a special case of Theorem 4.7 and formula (5.6.5) from [KR].

Proposition 6.2. *Let $L(\Lambda, c)$ be the irreducible \hat{gl} -module with the highest weight $\Lambda = (\Lambda_j)_{j \in \mathbb{Z}}$ and central charge k . Let $n_j = \Lambda_j - \Lambda_{j+1} + \delta_{j,0}c$ be the labels of Λ . Then when restricted to $\hat{\phi}_s(\hat{\mathcal{D}})$ the module $L(\Lambda, c)$ becomes an irreducible quasifinite $\hat{\mathcal{D}}$ -module with exponents $s-j$ ($j \in \mathbb{Z}$) of multiplicity n_j (and central charge c).*

Recall now that the $\hat{\mathcal{D}}$ -module $L(0, c; \hat{\mathcal{D}})$ has a canonical structure of a vertex algebra with the vacuum vector $|0\rangle = v_0$ and generated by the fields $J^k(z) = \sum_{m \in \mathbb{Z}} J_m^k z^{-m-k-1}$ [FKRW]. The following proposition is immediate from (5.4), (5.7) and (6.8).

Proposition 6.3. *We have an isomorphism of vertex algebras:*

$$L(0, -N; \hat{\mathcal{D}}) \simeq W_{1+\infty, -N}$$

under which the fields (denoted by the same symbol) $J^k(z)$ correspond to each other.

A $L(0, c; \hat{\mathcal{D}})$ -module M is called a positive energy module if the operator J_0^{1M} is diagonalizable and the set of real parts of its eigenvalues is bounded below. It is

clear that the irreducible positive energy $L(0, c; \hat{\mathcal{D}})$ -modules are some of the modules $L(\lambda, c; \hat{\mathcal{D}})$, and the problem is which of the $\Delta_\lambda(x)$ may occur. As was pointed out in [FKRW], all of them occur if $c \notin \mathbb{Z}$. One of the main results of [FKRW] is that the irreducible positive energy $L(0, c; \hat{\mathcal{D}})$ -modules with $c \in \mathbb{Z}_+$ are precisely the primitive modules with non-negative multiplicities of exponents.

We address now this problem in the remaining case $c = -N$, where N is a positive integer.

Theorem 6.1. *Viewed as a $W_{1+\infty, -N}$ -module the module M_s decomposes in a direct sum (with multiplicities) of all primitive modules for which the set of exponents lies in $s + \mathbb{Z}$, the difference of any two exponents does not exceed N and the multiplicity of only one exponent is negative.*

Proof. The proof follows from remarks made at the end of Section 4, Theorem 3.1, and Proposition 6.2. \square

Theorem 6.1 along with Proposition 6.1 imply

Theorem 6.2. *Let R be the set of exponents of a primitive $\hat{\mathcal{D}}$ -module V with central charge $-N$. Let s_1, s_2, \dots be all exponents with negative multiplicity. Assume that*

(i) $s_i - s_j \notin \mathbb{Z}$ if $i \neq j$.

Let $R_i = \{r \in R \mid r - s_i \in \mathbb{Z}\}$. Assume that

(ii) $R = \bigcup_i R_i$.

Let $N_i = -$ (sum of multiplicities of all exponents from R_i). Assume that

(iii) *difference of any two exponents from R_i does not exceed N_i .*

Then V is a $W_{1+\infty, -N}$ -module.

Conjecture 6.1. *Theorem 6.2 lists all irreducible $W_{1+\infty, -N}$ -modules.*

7. CHARGED FREE FERMIONS

In this section we show how to recover the main results of the paper [FKRW] (and to obtain some new results along the way) using the method of this paper applied to N pairs of free charged fermionic fields

$$\psi^i(z) = \sum_{m \in \mathbb{Z}} \psi_m^i z^{-m-1} \quad \text{and} \quad \psi^{*i}(z) = \sum_{m \in \mathbb{Z}} \psi_m^{*i} z^{-m}.$$

Recall that this is a collection of local odd fields with the OPE (2.1) and the vacuum vector $|0\rangle$ subject to conditions (2.2). Here and further we substitute ψ in place of γ . In other words, we have a unital associative algebra C_N , called the Clifford algebra, on generators ψ_m^i, ψ_m^{*i} ($i = 1, \dots, N; m \in \mathbb{Z}$) with the following defining relations (cf. (2.3)):

$$[\psi_m^{*i}, \psi_n^j]_+ \equiv \psi_m^{*i} \psi_n^j + \psi_n^j \psi_m^{*i} = \delta_{ij} \delta_{m, -n},$$

and all other anticommutators equal zero. The algebra C_N has a unique irreducible representation in a vector space F , called the spin representation, such that there exists a non-zero vector $|0\rangle$ satisfying (2.2). An important difference with the bosonic case is that F is a superspace with the parity

$$p(|0\rangle) = \bar{0}, \quad p(\psi_m^i) = p(\psi_m^{*i}) = \bar{1}.$$

We introduce the fields $e^{ij}(z)$ and $E(z, w)$ by the same formulas as in Section 2. Then we have to change the sign in the last summand of (2.4), meaning that the e_m^{ij} form a representation in F of the affine Kac-Moody algebra of level 1. Formulas (2.5) and (2.6) remain unchanged. We have to change the sign in the last summand of (2.7), meaning that the operators E_{ij} form a representation in F of the Lie algebra \widehat{gl} with central charge N .

In the same way as in Section 2, we prove

Proposition 7.1. (a) *The associative algebra $(C_N)^{gl_N}$ is generated by the elements E_{ij} ($i, j \in \mathbb{Z}$).*

(b) *As a gl_N -module, F is isomorphic to the exterior algebra over the gl_N -module $U_-^* \oplus U_+$, hence, in particular, decomposes into a direct sum of irreducible finite-dimensional gl_N -modules.*

(c) *Each gl_N isotypic component F_E is an irreducible $gl_N \oplus \widehat{gl}$ -module.*

Denote by H the set of sequences $(\lambda_i)_{i \in \mathbb{Z}}$ such that $(\lambda_i)_{i > 0} \in H_+$ and $(\lambda_i)_{i \leq 0} \in H_-$. Define a map $T : H \rightarrow H$ by:

$$(\lambda^T)_i = \#\{j | \lambda_j \geq i\} \text{ for } i > 0, \quad (\lambda^T)_i = -\#\{j | \lambda_j \leq i - 1\} \text{ for } i \leq 0.$$

In other words, when restricted to H_- and to H_+ , T is the usual transposition of the Young diagram with respect to the main diagonal.

Lemma 7.1. *The $gl_N \oplus gl_\pm$ -module $\Lambda(\mathbb{C}^{\pm N} \otimes \mathbb{C}^{\pm \infty})$ has the following decomposition into a direct sum of irreducible modules:*

$$\Lambda(\mathbb{C}^{\pm N} \otimes \mathbb{C}^{\pm \infty}) = \bigoplus_{\lambda \in H_N^\pm} (L_N(\lambda) \otimes L_\pm(\lambda^T)).$$

Proof. The proof follows from the other of Cauchy's formulae [M]:

$$\prod_{i=1}^N \prod_{j=1}^{\infty} (1 + x_i y_j) = \sum_{\lambda \in H_N^+} \text{ch } L_N(\lambda) \text{ch } L_+(\lambda^T). \quad \square$$

In the same way as in Section 2 we derive now the following result.

Proposition 7.2. *The following is a decomposition of F as a \mathfrak{g} -module:*

$$F = \bigoplus_{\substack{\lambda \in H_N^- \\ \mu \in H_N^+ \\ \nu \in H_N}} c_{\lambda, \mu}^\nu L_N(\nu) \otimes L_-(\lambda^T) \otimes L_+(\mu^T).$$

We are in a position now to prove the main result of this section.

Theorem 7.1. *With respect to $gl_N \oplus \widehat{gl}$, the spin representation F decomposes as follows:*

$$(7.1) \quad F = \bigoplus_{\nu \in H_N} L_N(\nu) \otimes L(\Lambda(\nu)^T, N),$$

where $\Lambda(\nu)$ is defined by (3.5).

It is clear that the proof of this theorem reduces to the following lemma.

Lemma 7.2. *For any $(\lambda; \mu) \in S_\nu$ (where S_ν is defined in Section 3) we have*

$$\Lambda(\nu)^T \geq (\lambda^T; \mu^T).$$

Proof. Define the following operations on the set of pairs $(\lambda; \mu) \in H_N^- \times H_N^+$:

$$\begin{aligned} T_{ij}^{(1)} : & \lambda_i \rightarrow \lambda_i - 1, \quad \lambda_j \rightarrow \lambda_j + 1 \text{ for } i < j, \\ T_{ij}^{(2)} : & \mu_i \rightarrow \mu_i - 1, \quad \mu_j \rightarrow \mu_j + 1 \text{ for } i < j, \\ T_{ij}^{(3)} : & \lambda_i \rightarrow \lambda_i + 1, \quad \mu_j \rightarrow \mu_j - 1 \text{ for any } i, j. \end{aligned}$$

We will say that $(\lambda'; \mu') \succ (\lambda; \mu)$ if one gets $(\lambda'; \mu')$ from $(\lambda; \mu)$ by a sequence of operations $T_{ij}^{(a)}$, $a = 1, 2, 3$.

It is straightforward to show

$$(7.2) \quad (\lambda'; \mu') \succ (\lambda; \mu) \text{ iff } (\lambda'^T; \mu'^T) > (\lambda^T; \mu^T),$$

where $>$ is the usual partial order on the set of weights of \widehat{gl} .

Fix $\nu \in H_N$ and define p as in (3.5). Let $r \in [1, p]$, $s \in [p+1, N]$. Recall that by Lemma 3.3 we have:

$$\mu_r \geq \nu_r, \quad \lambda_s \leq \nu_s.$$

Applying $T_{rs}^{(1)}$ for s such that $\lambda_s < \nu_s$ and any r sufficiently many times to $(\lambda; \mu)$ we shall get $(\lambda'; \mu')$ such that $\lambda'_s = \nu_s$ for each s . Similarly, applying $T_{rs}^{(2)}$ for r such that $\mu_r > \nu_r$, we shall get $(\lambda''; \mu'')$ such that $\lambda''_s = \nu''_s$ and $\mu''_r = \nu''_r$ for each r and s . It is clear that $q''_r \geq 0$ and $q''_s \leq 0$ and $\sum_i q''_i = 0$ for $(\lambda''; \mu'')$. Applying $T_{r,s}^{(3)}$ for non-zero q_r and q_s we get $(\tilde{\lambda}; \tilde{\mu})$ with the $\tilde{q}_i = 0$. It is clear that $(\tilde{\lambda}; \tilde{\mu}) \in S_\nu$ and $(\tilde{\lambda}; \tilde{\mu}) \succ (\lambda; \mu)$. Thus, we obtain that $\Lambda(\nu) \succ (\lambda; \mu)$. This together with (7.2) proves Lemma 7.2. \square

Remark 7.1. It is easy to see that decomposition (7.1) coincides with the decomposition [FKRW, (6.3)].

The first consequence of Theorem 7.1 is one of the main results of [FKRW]:

$$(7.3) \quad F^{gl_N} \simeq L(0, N) \text{ as } \widehat{gl}\text{-modules.}$$

As in Section 3, another corollary of Theorem 7.1 and Proposition 7.2 is a character formula:

$$(7.4) \quad \text{tr}_{L(\Lambda, N)} \text{diag}(\dots, y_2^{-1}, y_1^{-1}; x_1, x_2, \dots) = \sum_{\lambda \in H_N^+} c_{\lambda^* \mu}^\nu S_{\lambda^T}(y) S_{\mu^T}(x),$$

a special case of which is the following formula stated in [AFMO]:

$$(7.5) \quad \text{tr}_{L(0, N)} \text{diag}(\dots, y_2^{-1}, y_1^{-1}; x_1, x_2, \dots) = \sum_{\lambda \in H_N^+} S_{\lambda^T}(y) S_{\lambda^T}(x).$$

These formulas look quite different from [FKRW, (2.7)].

Remark 7.2. The \widehat{gl} -modules that occur in (7.1) are precisely all the modules with central charge N and dominant integral highest weights Λ (i.e., $\Lambda_i \in \mathbb{Z}$ and $\Lambda_i \geq \Lambda_j$ if $i \leq j$). Hence we have a complete reducibility theorem analogous to Theorem 4.1.

Remark 7.3. Consider the antilinear anti-involution of the algebra C_N defined by

$$(\psi_m^i)^\dagger = \psi_{-m}^{*i}.$$

The unique Hermitean form on F defined as in Remark 4.1 is positive definite. The anti-involution \dagger induces the usual compact anti-involutions ($A \rightarrow {}^t\bar{A}$) on gl_N and \widehat{gl} , and the anti-involution σ on $\widehat{\mathcal{D}}$ (see [KR]).

The space F of the spin representations carries a canonical structure of a vertex superalgebra defined in the same way as in Section 5 for M . The Γ_s -twisted F -modules F_s are defined in the same way too. The fields $J^k(z)$ defined in the same way as in Section 5, generate a vertex subalgebra of the vertex algebra F denoted by $W_{1+\infty, N}$. Due to (7.3), we have:

$$(7.6) \quad W_{1+\infty, N} = L^{gl_N}.$$

We also have a formula similar to (5.5) except for the change of sign of the last summand.

In the same way as in Section 6, we have an isomorphism of vertex algebras [FKRW]:

$$(7.7) \quad L(0, N; \widehat{\mathcal{D}}) \simeq W_{1+\infty, N}.$$

In the same way as in Section 6, we prove the following theorem. (Remark 5.2 of [FKRW] should be corrected accordingly.)

Theorem 7.2. *Viewed as a $W_{1+\infty, N}$ -module, the F -module F_s decomposes in a direct sum (with multiplicities) of all primitive modules for which the set of exponents lies in $s + \mathbb{Z}$ and multiplicities of all exponents are positive.*

Using Proposition 6.2 and Theorem 7.2, we see that all primitive modules whose exponents have positive multiplicities are $W_{1+\infty, N}$ -modules. It is shown in [FKRW] that these are all irreducible $W_{1+\infty, N}$ -modules. In particular, due to Remark 7.2, any positive energy module of the vertex algebra $W_{1+\infty, N}$ is completely reducible. Of course a similar statement for $W_{1+\infty, -N}$ follows from Conjecture 6.1.

REFERENCES

- [AFMO] H. Awata, M. Fukuma, Y. Matsuo, and S. Otake, *Character and determinant formulae of quasifinite representations of the $W_{1+\infty}$ algebra*, preprint, RIMS-982 (1994).
- [ABMN] L. Alvarez-Gaumé, J.B. Bost, G. Moore, P. Nelson, and C. Vafa, *Bosonization in arbitrary genus*, Phys. Lett. B **178** (1986), 41–45.
- [ASM] M. Adler, T. Shiota, and P. van Moerbeke, *From the w_∞ algebra to its central extension: A τ -function approach.*, preprint.
- [BMP] P. Bouwknegt, J. McCarthy, and K. Pilch, *Semi-infinite cohomology and w -gravity*, hep-th 9302086.
- [B1] R. Borcherds, *Vertex algebras, Kač-Moody algebras, and the Monster*, Proc. Natl. Acad. Sci. USA **83** (1986), 3068–3071.
- [B2] R. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), 405–444.

- [CTZ1] A. Capelli, C.A. Trugenberger, and G.R. Zemba, *Classifications of quantum Hall universality classes by $W_{1+\infty}$ symmetry*, Phys. Rev. Let. **72** (1994), 1902–1905.
- [CTZ2] A. Capelli, C.A. Trugenberger, and G.R. Zemba, *Stable hierarchical quantum Hall fluids as $W_{1+\infty}$ minimal models*, preprint, UGVA-DPT 1995/01–879.
- [D] J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars, Paris, 1974.
- [F] B.L. Feigin, *The Lie algebra $gl(\lambda)$ and the cohomology of the Lie algebra of differential operators*, Uspechi Math. Nauk **35**, no. 2 (1988), 157–158.
- [FKRW] E. Frenkel, V. Kac, A. Radul, and W. Wang, *$W_{1+\infty}$ and $W(gl_N)$ with central charge N* , Comm. Math. Phys. **170** (1995), 337–357.
- [FLM] I.B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, New York, 1988.
- [G] P. Goddard, *Meromorphic conformal field theory*, in Infinite-dimensional Lie algebras and groups (V. Kac, ed.); Adv. Ser. in Math. Phys. **7** (1989), World Scientific, 556–587.
- [H1] R. Howe, *Dual pairs in physics: harmonic oscillators, photons, electrons, and singletons*, Lectures in Appl. Math., vol. 21, 1985, pp. 179–206.
- [H2] R. Howe, *Remarks on classical invariant theory*, Trans. AMS **313** (1989), 539–570.
- [K1] V.G. Kac, *Infinite-dimensional Lie algebras, third edition*, Cambridge University Press, 1990.
- [K2] V.G. Kac, *Vertex algebras*, preprint.
- [KP] V.G. Kac and D.H. Peterson, *Spin and wedge representations of infinite-dimensional Lie algebras and groups*, Proc. Natl. Acad. Sci. USA **78** (1981), 3308–3312.
- [KR] V.G. Kac and A. Radul, *Quasi-finite highest weight modules over the Lie algebra of differential operators on the circle*, Comm. Math. Phys. **157** (1993), 429–457.
- [KV] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), 1–47.
- [L] S. Lang, *Algebra*, Addison-Wesley Publ. Co., Reading, MA, 1965.
- [Li] H. Li, *2-cocycles on the algebra of differential operators*, J. Algebra **122** (1989), 64–80.
- [M] I. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford, 1979.
- [Mat] Y. Matsuo, *Free fields and quasi-finite representations of $W_{1+\infty}$* , Phys. Lett. B. **326** (1994), 95–100.
- [O] G.I. Olshanskii, *Description of the representations of $U(p, q)$ with highest weight*, Funct. Anal. Appl **14** (1980), 32–44.
- [VP] E.B. Vinberg and V.L. Popov, *Invariant theory*, Itogi nauki i tekhniki, vol. 55, 1989, pp. 137–309.
- [Z] D.P. Zhelobenko, *Compact Lie groups and their representations*, Nauka, Moscow, 1970.

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